Fibrations

Nice class of maps, useful to compute htp grs.

Main application: knowing $\pi_n S^n \cong \mathbb{Z} \Rightarrow \pi_3 S^2 \cong \mathbb{Z}$

$\pi_m S^n$, $n \leq m$ are extremely interesting and complicated

Def. A cont map $p: E \to B$ is called a Serre fibration if for every commutative diagram

$$\begin{tikzcd} D^n \times \{0\} \arrow[r, "f"] \arrow[d, hook, "id \times incl"'] & E \arrow[d, "p"] \\ D^n \times I \arrow[r, "G"] & B \end{tikzcd} \quad \forall n \geq 0$$

there exists a cont. map $H: D^n \times I \to E$ such that the resulting diagram commutes:

$$\begin{tikzcd} D^n \times \{0\} \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ D^n \times I \arrow[r, "G"] \arrow[ur, dashed, "H"] & B \end{tikzcd} \quad (*) \quad \forall n \geq 0$$

i.e. $p \circ H = G$ and $f = H \circ (id \times incl)$. The morphism H is called a homotopy lifting, the diagram $(*)$ is called the homotopy lifting property of p. (Note that H need not be unique.)

Def. A Hurewicz fibration is a map $p: E \to B$ which satisfies the homotopy lifting property

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "G"] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

for every space X.

Clearly Hurewicz $\Rightarrow$ Serre.

Ex. The projection $F \times B \twoheadrightarrow B$ is a Hurewicz fibration.

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d, hook] & F \times B \arrow[d] \\ X \times I \arrow[r, "G"] \arrow[ur, dashed, "\exists ? \, H"] & B \end{tikzcd}$$

$H = (H_1, H_2)$, $f = (f_1, f_2)$

$H_1(x,t) := f_1(x) \quad \forall t$

$H_2(x,t) := G(x,t) \quad \longrightarrow$ the diag. commutes.

Ex. Serre fibration which is not Hurewicz

$A := \{0, 1, \dots\} = \mathbb{N}_0$ with the discrete topology

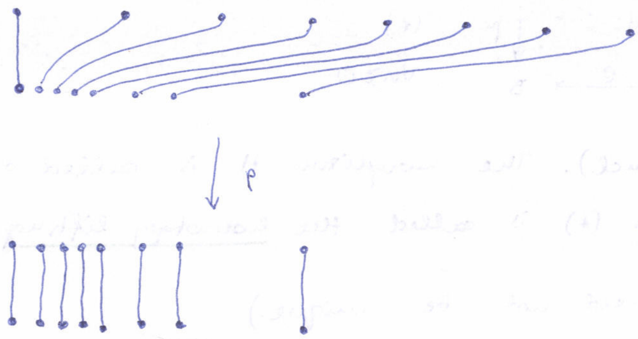
$B := \{0\} \cup \{\frac{1}{n} \mid n \geq 1\} \subseteq \mathbb{R}$ with the subspace topology (every point except 0 is discrete)

$$\begin{aligned} \alpha : A &\longrightarrow B \quad \text{continuous map} \\ 0 &\longmapsto 0 \\ \tfrac{1}{n} &\longmapsto n \end{aligned}$$

Take the pushout

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, hook] & B \arrow[d] \arrow[ddr, bend left, "\mathrm{incl}"] & \\ A \times I \arrow[r] \arrow[drr, bend right, "\alpha \times \mathrm{id}"'] & A \times I \coprod_\alpha B =: X \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists! p"] & \\ & & B \times I \end{tikzcd}$$

We claim that the morphism p constructed above is Serre but not Hurewicz.



$B \times I$

$(\frac{1}{n}, 0) \xrightarrow{n \to \infty} (0, 0)$

(n, t) does not converge to anything as $n \to \infty$

<u>Claim 1.</u> p is not a Hurewicz fibration.

$$\begin{tikzcd} B \times \{0\} \arrow[r] \arrow[d, hook] & A \times I \coprod_\alpha B \arrow[d, "p"] \\ B \times I \arrow[r, equal] \arrow[ur, dashed, "\cancel{H}"] & B \times I \end{tikzcd}$$

If there was a lift H, it must be the inverse of p since p is a bijection. But the inverse of p is not continuous. ↯

<u>Claim 2.</u> p is a Serre fibration.

Take a path-connected space C (e.g. $C = D^n$).

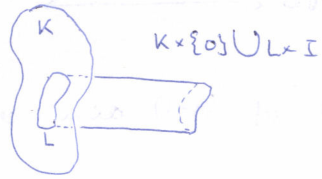
$$\begin{tikzcd} C \arrow[r] \arrow[d, hook] & A \times I \coprod_\alpha B \arrow[d, "p"] \\ C \times I \arrow[r] \arrow[ur, dashed, "H"] & B \times I \end{tikzcd}$$

Since a cont map sends path-connected spaces to path-connected ones, the bottom horizontal arrow sends $C \times I$ into one of the $| = \{b\} \times I \subsetneq B \times I$. Since $p|_{\{n\} \times I}$ is a homeo, let $H = (p|_I)^{-1} \circ G$. This makes the diagram commute.

Note that this example was somewhat artificial: most Serre fibrations one encounters are also Hurewicz fibrations.

<u>Lemma.</u> $E \xrightarrow{p} B$ a Serre fibration, K a CW-complex, $L \subseteq K$ a subcomplex. Then p has the following homotopy lifting property:

$$\begin{tikzcd} K \times \{0\} \cup L \times I \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ K \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists H"] & B \end{tikzcd}$$



<u>Cor.</u> A Serre fibration has the homotopy lifting property

$$\begin{tikzcd} K \times \{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ K \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

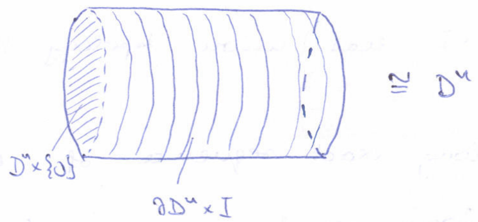
for every CW-complex K.

(Take $L = \emptyset$.)

Hence the difference b/w Hurewicz and Serre is taking all spaces / only CW-complexes.

PF OF LEMMA: First we observe that the lemma holds for the space $D^n \times \{0\} \cup \partial D^n \times I$, or in other words, for the pair $(D^n, \partial D^n)$.

$$\begin{tikzcd} D^n \arrow[r, phantom, "\cong"] \arrow[d, hook] & D^n \times \{0\} \cup \partial D^n \times I \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ D^n \times I \arrow[r, phantom, "\cong"] \arrow[urr, dashed, "H"] & D^n \times I \arrow[r, "F"'] & B \end{tikzcd}$$



The pf is induction on

$$C_n = K \times \{0\} \cup L \times I \cup K^{(n)} \times I \quad \text{where } K^{(n)} = sk_n K$$

$$\begin{tikzcd} K \times \{0\} \cup L \times I = C_{-1} \arrow[r, "f_{-1}"] \arrow[d, hook] & E \arrow[dddddd, "p"] \\ C_0 \arrow[ur, dashed] \arrow[d, hook] & \\ C_1 \arrow[uur, dashed] \arrow[d, hook] & \\ \vdots & \\ C_n \arrow[uuuur, dashed, "G_n"] \arrow[d, hook] & \\ \vdots & \\ \text{colimit: } K \times I \arrow[r, "F"'] & B \end{tikzcd}$$

We construct the dashed arrows inductively for $n \geq -1$:

$$G_{n-1} = G_n \circ \mathrm{incl}, \quad p \circ G_n = F_n \circ \mathrm{incl}$$

$n = -1$: $K^{(-1)} = \emptyset$

$$C_{-1} = K \times \{0\} \cup L \times I \xrightarrow{f =: G_{-1}} E$$

Suppose that G_n is already defined.

There is a pushout

$$\begin{tikzcd}
\coprod_{\alpha \in \mathcal{E}_{n+1}} \left(D^{n+1} \times \{0\} \cup \partial D^n \times I \right) \arrow[r] \arrow[d, hook] & C_n \arrow[d, hook] \\
\coprod_{\alpha \in \mathcal{E}_{n+1}} \left(D^{n+1} \times I \right) \arrow[r] & C_{n+1} \arrow[ul, phantom, "\ulcorner" very near start]
\end{tikzcd}$$

where $\mathcal{E}_{n+1}$ is the set of $(n+1)$-dim cells of K which don't lie in L.

$$\begin{tikzcd}
\coprod \left(D^{n+1} \times \{0\} \cup \partial D^n \times I \right) \arrow[r] \arrow[d, hook] & C_n \arrow[r, "G_n"] \arrow[d, hook] & E \arrow[d, "p"] \arrow[ddl, bend left, equal] \\
\coprod \left(D^{n+1} \times I \right) \arrow[r] \arrow[urr, dashed, "\Psi"] \arrow[dr, "\Psi"'] & C_{n+1} \arrow[r] \arrow[d, dashed, "\exists ! G_{n+1}"] \arrow[ul, phantom, "\ulcorner" very near start] & B \\
 & E &
\end{tikzcd}$$

Ψ exists by the previous step. Use the pushout to get G_{n+1}.

By construction G_{n+1} satisfies the required properties.

$K \times I$ has union topology $K \times I = \bigcup_{n \geq -1} C_n \xrightarrow{\bigcup_{n \geq -1} G_n} E$ $\Rightarrow$ this is a htp lifting. □

The long exact sequence of homotopy groups

Let $p : E \longrightarrow B$ be a Serre fibration, $b_0 \in B$ a point.

The fiber of p at b_0 is $F := p^{-1}(b_0) = \{ e \in E \mid p(e) = b_0 \}$.

We assume F to be non-empty: although p could be non-surjective, in practice it will be. Choose $e_0 \in F$.

We get a sequence of pointed maps

$$(F, e_0) \overset{\iota}{\hookrightarrow} (E, e_0) \xrightarrow{p} (B, b_0)$$

Prop. There is a long exact sequence

$$\pi_n(F, e_0) \xrightarrow{\iota_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0) \xrightarrow{\iota_*} \pi_{n-1}(E, e_0) \xrightarrow{p_*} \cdots$$

$$\cdots \xrightarrow{\partial} \pi_1(F, e_0) \xrightarrow{\iota_*} \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial} \pi_0(F, e_0) \xrightarrow{\iota_*} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(B, b_0)$$

Rmk. Exactness at the end is understood in the category of pointed sets, not of groups. (since there is no group structure).

Pf: The assertion could be deduced from the les for pairs by showing $\pi_n(B,b_0)\cong\pi_n(E,F)$. We will prove it directly instead.

Convention: $\pi_n(X,x_0)=[(I^n,\partial I^n),(X,x_0)]$

We define $\partial:\pi_{n+1}(B,b_0)\longrightarrow\pi_n(F,e_0)$.

Let $[\varphi]\in\pi_{n+1}(B,b_0)$. The diagram commutes since $\varphi(\partial I^{n+1})=b_0$

$$\begin{tikzcd}
I^n\times\{0\}\cup\partial I^n\times I \arrow[r, "\mathrm{const}_{e_0}"] \arrow[d, hook] & E \arrow[d, "p"] \\
I^n\times I \arrow[r] \arrow[ur, dashed, "\Phi"] & B
\end{tikzcd}$$

Let $\partial[\varphi]:=[\Phi(-,1)]$.

Well-definedness:

1) $\Phi(t,1)\in F\ \forall t\in I^n$ since $p\Phi(t,1)=\varphi(\underbrace{t,1}_{\in\partial I^{n+1}})=b_0$

2) Based map: $\Phi(t,1)=e_0$ if $\partial I^n\ni t$ by the upper triangle.

3) Independence of representatives: let $H: I^{n+1}\times I\longrightarrow B$, $H(-,0)=\varphi$, $H(-,1)=\varphi'$, $H(\partial I^{n+1},-)=b_0$. Choose lifts:

$$\begin{tikzcd}
I^n\cup\partial I^n\times I \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\
I^n\times I \arrow[r, "{\varphi,\ \varphi'}"'] \arrow[ur, dashed, "{\Phi,\ \Phi'}"] & B
\end{tikzcd}$$

WTS $\Phi(-,1)\simeq\Phi'(-,1)$.

There is a CW-inclusion $(I^n\times\{0\}\cup\underbrace{I^n\times\{1\}}_{L}\cup\partial I^n\times I)\subseteq\underbrace{I^n\times I}_{K}$

$\Rightarrow$ there is a lift:

$$\begin{tikzcd}
\overbrace{I^n\times I\times\{0\}}^{e_0}\cup\overbrace{I^n\times\{1\}\times I}^{\Phi}\cup\overbrace{I^n\times\{1\}\times I}^{\Phi'}\cup\overbrace{\partial I^n\times I\times I}^{e_0} \arrow[r, "\mathcal{H}"] \arrow[d, hook] & E \arrow[d, "p"] \\
I^n\times I\times I \arrow[r, "\hat{H}"] \arrow[ur, dashed, "G"] & B
\end{tikzcd}$$

$\hat{H}(x,r,s)=H(x,s,r)$

$g:=G(-,1)$ is the needed homotopy.

$\Rightarrow\partial$ is well-defined. ✓

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We show that ∂ is a homomorphism.

$$\begin{tikzcd} I^n \times \{0\} \cup \partial I^n \times I \arrow[r, "e_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "\varphi"] \arrow[ur, dashed, "\exists \Phi"] & B \end{tikzcd} \qquad \partial[\varphi] := [\Phi(-,1)] \in \pi_n F$$

e_0 is a lift for $b_0 \Rightarrow \partial[b_0] = e_0$, thus ∂ preserves the unit

Let $[\varphi], [\varphi'] \in \pi_{n+1} B$. Then $[\varphi]+[\varphi'] = \left[(t_1,\dots,t_{n+1}) \mapsto \begin{cases} \varphi(2t_1, t_2, \dots, t_{n+1}) & t_1 \in [0, 1/2] \\ \varphi(2t_1-1, t_2, \dots, t_{n+1}) & t_1 \in [1/2, 1] \end{cases} \right]$

Note that the last coordinate t_{n+1} always remains untouched.

Choose lifts Φ, Φ' for φ, φ' and obtain

$$\begin{tikzcd} I^n \times \{0\} \cup \partial I^n \times I \arrow[r, "e_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "\varphi+\varphi'"'] \arrow[ur, "\Phi+\Phi'"'] & B \end{tikzcd}$$

$$\Rightarrow \partial([\varphi]+[\varphi']) = \partial[\varphi+\varphi'] = [(\Phi+\Phi')(-,1)] = [\Phi(-,1)] + [\Phi'(-,1)]$$
$$= \partial[\varphi] + \partial[\varphi'],$$

so ∂ is indeed a homomorphism. ✓

It remains to show exactness.

We prove exactness at $\pi_{n+1} B$:

Im $p_* \subseteq$ Ker ∂: let $[\alpha] \in \pi_{n+1} F$. Then $\partial p_*[\alpha] = \partial[p \circ \alpha]$.

We have α as a lift:

$$\begin{tikzcd} I^n \times \{0\} \cup \partial I^n \times I \arrow[r, "e_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "p \circ \alpha"'] \arrow[ur, "\alpha"] & B \end{tikzcd}$$

i.e. $\partial[p \circ \alpha] = [\underbrace{\alpha(-,1)}_{\in \partial I^{n+1}}] = [e_0]$.

$\Rightarrow [p \circ \alpha] \in$ Ker ∂. ✓

Ker $\partial \subseteq$ Im p_*: let $[\varphi] \in$ Ker ∂.

We have the commutative diagram

$$\begin{tikzcd} I^n \times \{0\} \cup \partial I^n \times I \arrow[r, "e_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "\varphi"'] \arrow[ur, dashed, "\Phi"'] & B \end{tikzcd}$$

We also have a relative homotopy $G : I^n \times I \longrightarrow F$,

$$G(t,0) = \Phi(t,1),$$
$$G(t,1) = e_0,$$
$$G(\partial I^n, -) = e_0 \quad \text{(this means that } G \text{ is rel.)}$$

Consider the diagram

$$\begin{tikzcd}
\Big(\overbrace{I^n \times I \times \{0\}}^{\Phi}\Big) \cup \Big(\overbrace{I^n \times \{0\} \times I}^{e_0}\Big) \cup \Big(\overbrace{I^n \times \{1\} \times I}^{G}\Big) \cup \Big(\overbrace{\partial I^n \times I \times I}^{e_0}\Big) \arrow[r] \arrow[d, hook] & E \arrow[d, "\text{fibration}"] \\
I^n \times I \times I \arrow[r] \arrow[ur, dashed, "H"] & *
\end{tikzcd}$$

where H exists by HEP.

$\Rightarrow$ $p \circ H(t_1, \dots, t_n, s, -)$ is a htp from

$$p \circ H(t_1, \dots, t_n, s, 0) = p \circ \Phi(t_1, \dots, t_n, s) = \varphi(t_1, \dots, t_n, s)$$

to $p \circ H(t_1, \dots, t_n, s, 1) = h(t_1, \dots, t_n, s)$

$\Rightarrow$ $p_*[h] = [p \circ h] = [\varphi]$

$$p \circ H(\partial I^{n+1}, u) = p \circ H(\partial I^n \times I \cup I^n \times \{0\} \cup I^n \times \{1\}) = p(e_0 \cup e_0 \cup G) = b_0$$

since $p \circ G = b_0$ b/c G targets the fibre F.

Exactness at $\pi_n F$ and $\pi_n E$ have (apparently) not been proven and probably have similar proofs anyway.

Fibre bundles

Every fibre bundle will be a fibration (to be proven next week) and will thus provide many examples.

Def. A map $p : E \longrightarrow B$ is a trivial fibration with fibre F when

$$\begin{tikzcd}
E \arrow[rr, "\exists \cong"] \arrow[dr, "p"'] & & B \times F \arrow[dl, "pr_B"] \\
& B \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

This gives a Hurewicz fibration but the les is uninteresting.

Def. A <u>fibre bundle with fibre F over B</u> is a map $p: E \longrightarrow B$

s.t. $\forall b \in B$ $\exists U \ni b$ open nbhd. and $\exists \tau$ iso making

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\tau", "\cong"'] \arrow[dr, "p|_{p^{-1}(U)}"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

commute, i.e. $p|_{p^{-1}(U)}$ is a trivial fibration.

In other words, p is a locally trivial fibration.

Ex. A covering space $p: E \longrightarrow B$ is a fibre bundle with discrete fibre if B is connected.

Recall that covering spaces have the HLP (with uniqueness if an initial condition is also given).

Ex. Hopf fibration. Recall the map $h: S^{2n+1} \longrightarrow \mathbb{C}P^n$, $S^{2n+1} \subset \mathbb{C}^{n+1}$

$$(z_0, \dots, z_n) \longmapsto [z_0, \dots, z_n]$$

h is a fibre bundle with S^1 as a fibre

Let $U_i := \{[z_0, \dots, z_n] \mid z_i \neq 0\} \subseteq \mathbb{C}P^n$ well-def'd open. and they cover $\mathbb{C}P^n$.

There is an iso

$$\begin{tikzcd} h^{-1}(U_i) \arrow[rr, "\tau", "\cong"'] \arrow[dr, "h"'] & & U_i \times S^1 \arrow[dl, "pr_{U_i}"] \\ & U_i & \end{tikzcd}$$

$n=1$: get $\eta: S^3 \longrightarrow \mathbb{C}P^1 \cong S^2$. The same can be done /$\mathbb{H}$ resp /$\mathbb{O}$ to obtain $S^7 \longrightarrow S^4$ resp $S^{15} \longrightarrow S^8$.

Lemma. The pullback of a fibration (Hurewicz) or a fibre bundle is a fibre or a fibre bundle with the same fibre.

i.e. $\forall f: X \longrightarrow B$ and $p: E \longrightarrow B$ fibration / fb. bundle the base change $\bar{p}: X \times_B E \longrightarrow X$ is a fiber / fb. bundle.

Pf: <u>For fibrations:</u>

$$\begin{tikzcd}
Z \arrow[r, "\beta"] \arrow[d, hook] & P \arrow[r] \arrow[d, "\bar p"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
Z\times I \arrow[r, "\alpha"'] \arrow[urr, dashed, "F"] & X \arrow[r, "f"'] & B
\end{tikzcd}$$

by HLP for p

That is,

$$\begin{tikzcd}
Z\times I \arrow[dr, dashed] \arrow[drr, bend left, "F"] \arrow[ddr, bend right, "\alpha"'] & & \\
& P \arrow[r] \arrow[d, "\bar p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
& X \arrow[r, "f"'] & B
\end{tikzcd}$$

by univ prop

Lift:

$$\begin{tikzcd}
Z \arrow[r, "\beta"] \arrow[d, hook, "i"'] & P \arrow[r] \arrow[d, "\bar p"] & E \arrow[d, "p"] \\
Z\times I \arrow[r, "\alpha"'] \arrow[ur, "F^*"'] \arrow[u, phantom, "*", near end, shift left=3] & X \arrow[r, "f"'] & B
\end{tikzcd}$$

$$\begin{tikzcd}
Z \arrow[dr, bend left, "\beta"] \arrow[dr, bend right, "\hat F \circ i"'] \arrow[drr, bend left=40] \arrow[ddr, bend right=40, "\alpha\circ i"'] & & \\
& P \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
& X \arrow[r, "f"'] & B
\end{tikzcd}$$

HLP for p

HLP for p

uniqueness $\Rightarrow$ $\hat F \circ i = \beta$

$\Rightarrow$ * commutes.

<u>For fibre bundles:</u> for $x \in X$ choose $V := p^{-1}(U)$, $U \ni f(x)$ and $p|_{p^{-1}(U)}$ trivial

Wts: $\bar p^{-1}(V) = \bar p^{-1}(f^{-1}(U)) \xrightarrow[\cong]{\tau} f^{-1}(U) \times F$

$$\begin{tikzcd}
\bar p^{-1}(f^{-1}(U)) \arrow[r, "\tau", "\cong"'] \arrow[dr] & f^{-1}(U)\times F \arrow[d] \\
& f^{-1}(U)
\end{tikzcd}$$

$\bar p^{-1}(V) = \{(x,e) \in X \times E \mid p(e) = f(x) \in U\}$

We can choose α for which

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\alpha", "\cong"'] \arrow[dr, "p"'] & & U\times F \arrow[dl, "pr_U"] \\
& U &
\end{tikzcd}$$

$\tau(x,e) := (x, pr_F(\alpha(e)))$, $\tau^{-1}(x,v) := (x, \alpha^{-1}(f(x),v))$ These are well-def'd.

And mutually inverse: $(x, \alpha^{-1}(f(x), pr_F\,\alpha(e))) = (x, \alpha^{-1}(p(e), pr_F\,\alpha(e))) = (x, \alpha^{-1}(\alpha(e))) = (x,e)$

$(x, pr_F \circ \alpha \circ \alpha^{-1}(f(x),v)) = (x, pr_F(f(x),v)) = (x,v)$